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Short Communication

Analytical solution for vibration of an incompressible isotropic linear elastic rectangular plate, and frequencies missed in previous solutions

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Abstract

An analytical solution is given for free vibration of a simply supported rectangular plate made of an incompressible linear elastic isotropic material. Through-the-thickness modes of vibration valid for compressible and incompressible materials and missed by previous investigators are also identified.

1. Introduction

Srinivas et al. [1] gave in 1970 an analytical solution for the vibration of simply supported homogeneous thick rectangular laminated plates. Since then the technique has been extended to piezoelectric and hybrid plates; e.g. see Refs. [2,3]. Batra and Aimmanee [4] recently pointed out some of the frequencies missed by Srinivas et al. [1] and others who employed the same approach. All these and several other investigations (e.g. see Refs. [5,6]) assume that the plate material is compressible. Because of the increasing use of rubberlike materials and the realization that many biological materials can be modeled as incompressible, we provide here an analytical solution for free vibration of a simply supported plate made of a homogeneous and isotropic incompressible linear elastic material. Of course, only isochoric (i.e., volume preserving) deformations are admissible in an incompressible body; thus it can undergo only pure distortional deformations. Corresponding to the constraint of incompressibility, the constitutive relation involves a hydrostatic pressure that cannot be determined from the deformation field but is found from a solution of the balance of linear momentum and traction (i.e., natural) boundary conditions prescribed at least on a part of the boundary of the body. For a displacement boundary-value problem the pressure field can be found only to within an arbitrary constant. For free vibrations of a plate the top and the bottom surfaces are traction free,

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and the pressure field is uniquely determined. The analytical solution provides benchmark frequencies for comparison with those computed from a plate theory. These are needed to ensure that frequencies obtained by taking the limit of results as Poisson's ratio approaches 0.5 are indeed correct. We also find additional frequencies missed in Ref. [4] and other previous analyses employing Srinivas et al.'s [1] technique.

For a simply supported square and a rectangular plate of aspect ratios (thickness/larger in-plane dimension) 1/4, 1/8, 1/12 and 1/20 we give numerical values of analytical frequencies.

2. Formulation of the problem

In rectangular Cartesian coordinates and in the absence of body forces infinitesimal deformations of an incompressible body are governed by the following balance of mass and the balance of linear momentum:

$$u_{i,i} = 0, \tag{1a}$$

$$\rho \ddot{u}_i = \sigma_{ij,j}, \tag{1b}$$

for $i, j = 1, 2, 3$.

Here $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{u}$ the displacement, $\rho > 0$ the mass density, a superimposed dot indicates differentiation with respect to time t, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$, $\mathbf{x}$ the present position of a material point, and a repeated index implies summation over the range of the index. Eq. (1a) implies that deformations are isochoric or volume preserving and hence the mass density stays constant. The balance of moment of momentum is identically satisfied by requiring that the stress tensor $\boldsymbol{\sigma}$ be symmetric. For an incompressible linear elastic isotropic material

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \tag{2a}$$

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \tag{2b}$$

where p is the hydrostatic pressure not determined from the infinitesimal strain tensor $\mathbf{e}$, δ_{ij} is the Kronecker delta, and $\mu > 0$ is the shear modulus. Substitution for $\mathbf{e}$ from Eq. (2b) into Eq. (2a) and for $\boldsymbol{\sigma}$ from Eq. (2a) into Eq. (1b) gives

$$\rho \ddot{u}_i = -p_{,i} + \mu u_{i,jj}. \tag{3}$$

Here we have assumed that the body is homogeneous; thus μ and ρ are constants.

For a simply supported rectangular plate occupying the region $[0, L_x] \times [0, L_y] \times [0, h]$, boundary conditions are listed below:

$$u_2 = u_3 = 0, \quad \sigma_{11} = 0 \text{ on } x_1 = 0,\ L_x; \tag{4a}$$

$$u_1 = u_3 = 0,\ \sigma_{22} = 0 \text{ on } x_2 = 0,\ L_y; \tag{4b}$$

$$\sigma_{i3} = 0 \text{ on } x_3 = 0,\ h. \tag{4c}$$

Thus the top and the bottom surfaces of the plate are traction free. The lateral deflection u_3 and the normal tractions vanish on all four edge surfaces. The boundary conditions given in Eq. (4) are not easily realized in a laboratory where the plate edges are typically supported on rollers or sharp knife wedges. However, they have been widely used since Srinivas et al. [1] presented analytical solutions for free vibrations of a rectangular plate made of a compressible linear elastic material. For the steady-state vibration problem no initial conditions are needed.

A difference between the problem for compressible and incompressible materials is that for the former unknowns are displacements but for the latter unknowns are displacements and the pressure field. The pressure field cannot be determined from a knowledge of displacements or strains and is to be found as a part of the solution of the problem. The pressure field can be found uniquely only if normal surface tractions are prescribed on a part of the boundary. Whereas all displacement fields are admissible in a body comprised of a compressible material, only isochoric deformations are admissible in a body made of an incompressible material.

3. Analytical solution

In order to solve for free vibrations of a simply supported rectangular plate we assume that Eqs. (1)–(4) have a solution of the form

$$u_1 = \sum_{m,n=0}^{\infty} U_1^{mn}(x_3)\cos Mx_1 \sin Nx_2 e^{i\omega t}, \tag{5a}$$

$$u_2 = \sum_{m,n=0}^{\infty} U_2^{mn}(x_3)\sin Mx_1 \cos Mx_2 e^{i\omega t}, \tag{5b}$$

$$u_3 = \sum_{m,n=0}^{\infty} U_3^{mn}(x_3)\sin Mx_1 \sin Nx_2 e^{i\omega t}, \tag{5c}$$

$$p = \sum_{m,n=0}^{\infty} P^{mn}(x_3)\sin Mx_1 \sin Nx_2 e^{i\omega t}, \tag{5d}$$

$$M = \frac{m\pi}{L_x}, \quad N = \frac{n\pi}{L_y}. \tag{5e,f}$$

Here m and n are integers, U_1^{mn}, U_2^{mn}, U_3^{mn} and P^{mn} are functions of x_3 that are to be determined, ω is a natural frequency and t is time. The form, Eqs. (5a)–(5c), of displacements is identical to that assumed by Srinivas et al. [1] except that we allow m and/or n to take the value zero whereas they did not, and we have an additional unknown, namely, the pressure field p. Here we have postulated a similar expression for the hydrostatic pressure p. Batra and Aimmanee [4] have pointed out that the lower limit in Eq. (5) ought to be 0 rather than 1 as has been assumed in numerous previous studies, e.g. see Refs. [2,3,7–9]. Assuming that the infinite series in Eq. (5) are uniformly convergent, we substitute for p and **u** from Eq. (5) into Eq. (3) and Eq. (1a) and obtain

$$\sum_{m,n=0}^{\infty}\{-MP^{mn} + (-\mu M^2 - \mu N^2 + \mu d_z^2 + \rho\omega^2)U_1^{mn}\}\cos Mx_1 \sin Nx_2 = 0,$$

$$\sum_{m,n=0}^{\infty}\{-NP^{mn} + (-\mu M^2 - \mu N^2 + \mu d_z^2 + \rho\omega^2)U_2^{mn}\}\sin Mx_1 \cos Nx_2 = 0,$$

$$\sum_{m,n=0}^{\infty}\{-d_z P^{mn} + (-\mu M^2 - \mu N^2 + \mu d_z^2 + \rho\omega^2)U_3^{mn}\}\sin Mx_1 \sin Nx_2 = 0,$$

$$\sum_{m,n=0}^{\infty}\{-MU_1^{mn} - NU_2^{mn} + d_z U_3^{mn}\}\sin Mx_1 \sin Nx_2 = 0 \tag{6}$$

and d_z and d_z^2 denote the first and the second derivative with respect to $z = x_3$.

Following the same procedure as that used in Ref. [1], we conclude that for each (m,n) combination the nontrivial solution of Eq. (6) is

$$\begin{Bmatrix} U_1^{mn} \\ U_2^{mn} \\ U_3^{mn} \\ P^{mn} \end{Bmatrix} = \begin{bmatrix} M & -M & fM & fM & N & N \\ N & -N & fN & fN & -M & -M \\ g & g & g^2 & -g^2 & 0 & 0 \\ \rho\omega^2 & -\rho\omega^2 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} C_1 e^{gz} \\ C_2 e^{-gz} \\ C_3 e^{fz} \\ C_4 e^{-fz} \\ C_5 e^{fz} \\ C_6 e^{-fz} \end{Bmatrix}, \tag{7}$$

where $C_1, C_2, \ldots, C_6$ are six arbitrary constants, and

$$f = \sqrt{g^2 - \frac{\rho\omega^2}{\mu}}, \quad g = \sqrt{M^2 + N^2}. \tag{8}$$

With the displacement vector **u** and the pressure p defined by Eq. (5), boundary conditions given in Eqs. (4a) and (4b) on edges $x_1 = 0,\ L_x$ and $x_2 = 0,\ L_y$ are identically satisfied. Furthermore, boundary conditions given in Eq. (4c) on the top and the bottom surfaces of the plate determine $C_1, C_2, \ldots, C_6$, and lead to the following characteristic equation:

$$[8g^3 f(g^2 + f^2)^2(1 - \cosh fh \cosh gh) + \{16g^6 f^2 + (g^2 + f^2)^4\} \sinh fh \sinh gh] f^2 \sinh fh = 0. \tag{9}$$

The term in square brackets is the same as that in Ref. [1] where the material of the plate was assumed to be compressible. Eq. (9) is satisfied when either

$$f = 0, \tag{10a}$$

or

$$\sinh(fh) = 0, \tag{10b}$$

or

$$8g^3 f(g^2 + f^2)^2(1 - \cosh fh \cosh gh) + \{16g^6 f^2 + (g^2 + f^2)^4\} \sinh(fh) \sinh(gh) = 0. \tag{11}$$

Eq. (10a) is a special case of Eq. (10b) since $f = 0$ is also a solution of Eq. (10b). For Eq. (10b) to be satisfied

$$f = \frac{i\alpha\pi}{h}, \quad \alpha = 0, 1, 2, \ldots, \tag{12}$$

in which case $C_1 = C_2 = 0$. The corresponding frequencies and mode-shapes are given by

$$\omega = \sqrt{\frac{\mu}{\rho}}\left(g^2 + \frac{\alpha^2\pi^2}{h^2}\right)^{1/2}, \quad \alpha = 0, 1, 2, \ldots, \tag{13}$$

$$\begin{Bmatrix} U_1^{mn} \\ U_2^{mn} \\ U_3^{mn} \\ P^{mn} \end{Bmatrix} = C_7 \begin{Bmatrix} -M \sin\left(\frac{\alpha\pi z}{h}\right) \\ -N \sin\left(\frac{\alpha\pi z}{h}\right) \\ \frac{\alpha\pi}{h} \cos\left(\frac{\alpha\pi z}{h}\right) \\ 0 \end{Bmatrix} + C_8 \begin{Bmatrix} N \cos\left(\frac{\alpha\pi z}{h}\right) \\ -M \cos\left(\frac{\alpha\pi z}{h}\right) \\ 0 \\ 0 \end{Bmatrix}, \tag{14}$$

Table 1
For $(m, n) = (1, 0), (1, 1), (2, 0)$, and $(2, 1)$ first-three natural frequencies and mode shapes of vibration of a thick square rubber plate with $h/L_x = 1/4$; (i) and (o) in column 2 following a value of the frequency indicate, respectively, the in-plane and the out-of-plane mode of vibration

(m, n)	Ω	$\{C_1, C_2, \ldots, C_6\}$
(1,0)	0.785 (i)	{0, 0, 0, 0, 0.707, 0.707}
	3.238 (i)	{0, 0, 0, 0, 0.707, 0,707}
	6.332 (i)	{0, 0, 0, 0, 0.707, 0.707}
(1,1)	0.577 (o)	{0.219, 0.666, −0.257, 0.665, 0, 0}
	1.111 (i)	{0, 0, 0, 0, 0.707, 0.707}
	2.111 (i)	{−0.263, 0.799, 0.299 + 0.238i, −0.299 + 0.238i, 0, 0}
(2,0)	1.571 (i)	{0, 0, 0, 0, 0.707, 0.707}
	3.512 (i)	{0, 0, 0, 0, 0.707, 0.707}
	6.477 (i)	{0, 0, 0, 0, 0.707, 0.707}
(2,1)	1.186 (o)	{−0.140, −0.810, 0.151, 0.550, 0, 0}
	1.756 (i)	{0, 0, 0, 0, 0.707, 0.707}
	3.099 (i)	{−0.145, 0.839, 0.355 + 0.107i, −0.355 + 0.107i, 0, 0}

where $C_7 \neq 0$ only when $g^2 + f^2 = 0$. C_7 and C_8 are arbitrary constants. These modes of vibration have been identified in Ref. [1] on the assumption that $\nu \neq 0.5$. The present analysis shows their validity even when $\nu = 0.5$. The modes with $C_7 = 0$ correspond to thickness modes in which displacement $u_3 = 0$. Whereas in Srinivas et al.'s solution [1] the first thickness mode corresponds to $\alpha = 1$ and $m = n = 1$, in our case it corresponds to $\alpha = 0$ and either $m = 1$, $n = 0$ or $m = 0$, $n = 1$ whichever has a lower frequency. If $m \neq 0$ and $n = 0$, then the frequency ω and the corresponding mode shapes are given by

$$\omega^2 = \frac{\mu}{\rho}\left(\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{\alpha\pi}{h}\right)^2\right), \quad (u_1, u_2, u_3) = \left(0,\ A\cos\frac{\alpha\pi z}{h}\sin\frac{m\pi x_1}{L_x},\ 0\right), \tag{15}$$

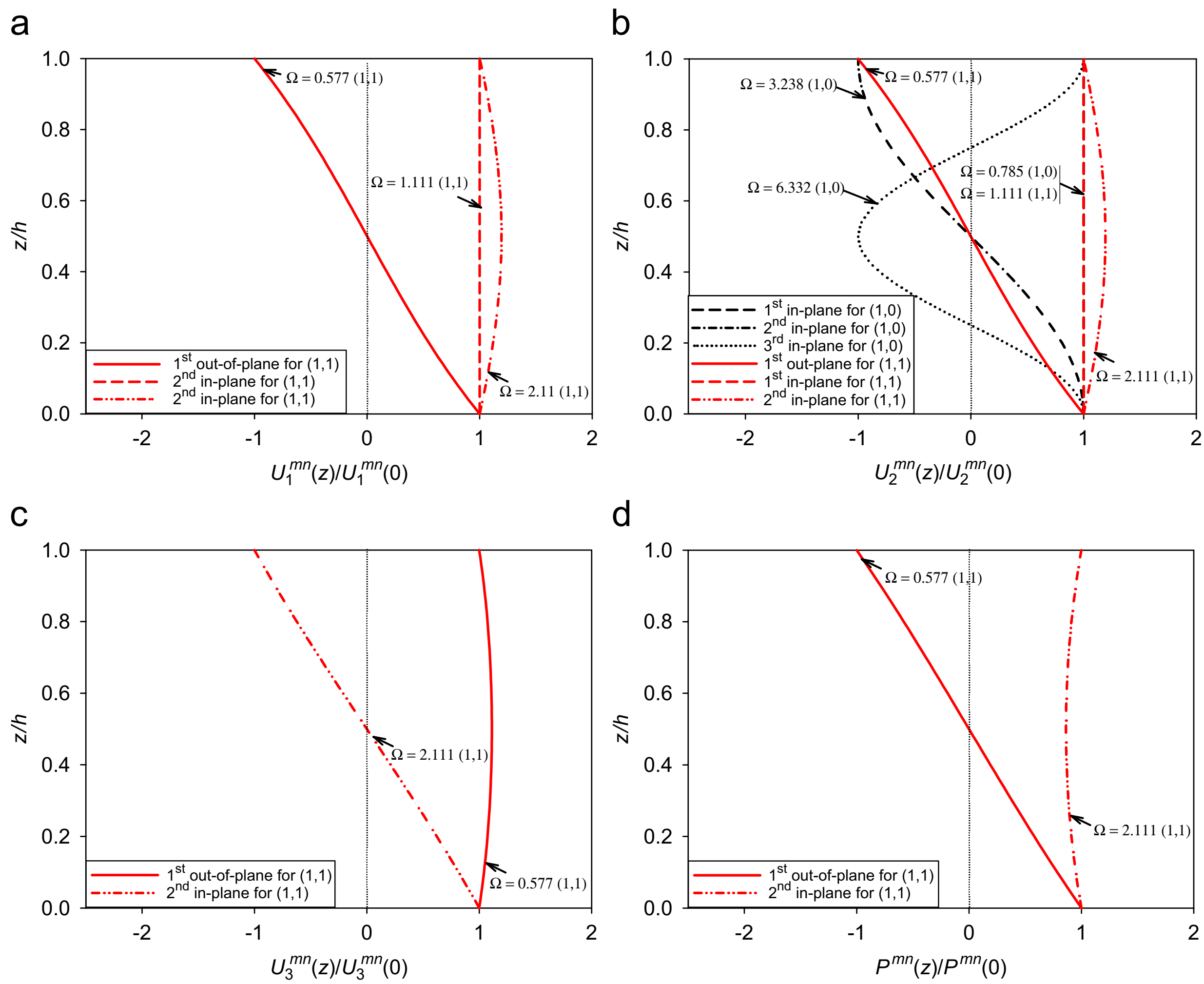

Fig. 1. For $(m,n) = (1,0)$ and $(1,1)$ through-the-thickness variation of the normalized displacement components and the pressure for three thickness modes of vibration. (a) ——1st out-of-plane for $(1,1)$, $\Omega = 0.577$; – – – –2nd in-plane for $(1,1)$, $\Omega = 1.11$; and –··–··–··2nd in-plane for $(1,1)$, $\Omega = 2.11$; (b) – – – – 1st in-plane for $(1,0)$, $\Omega = 3.238$; –· –·–·–2nd in-plane for $(1,0)$, $\Omega = 0.785$; ....3rd in-plane for $(1,0)$, $\Omega = 6.332$; ——1st out-of-plane for $(1,0)$, $\Omega = 0.577$; – – – –1st in-plane for $(1,0)$, $\Omega = 1.111$; and –··–··–··–2nd in-plane for $(1,0)$, $\Omega = 2.111$; (c) ——1st out-of-plane for $(1,1)$, $\Omega = 2.111$ and –··–··–·· 2nd in-plane for $(1,1)$, $\Omega = 0.577$; and (d) ——1st out-of-plane for $(1,1)$, $\Omega = 0.577$ and –··–··–··2nd in-plane for $(1,1)$, $\Omega = 2.111$.

where A is a constant. Similarly if $m = 0$ and $n \neq 0$, then the frequency ω and the corresponding mode shapes are given by

$$\omega^2 = \frac{\mu}{\rho}\left(\left(\frac{n\pi}{L_y}\right)^2 + \left(\frac{\alpha\pi}{h}\right)^2\right),\ (u_1, u_2, u_3) = \left(A\cos\frac{\alpha\pi z}{h}\sin\frac{n\pi x_2}{L_y},\ 0,\ 0\right). \tag{16}$$

These modes of vibration are also admissible in a simply supported rectangular plate made of a compressible material and were missed by Batra and Aimmanee [4], Srinivas et al. [1], and various other previous investigators. Batra and Aimmanee [4]'s solution corresponds to $\alpha = 0$ in Eqs. (15) and (16).

For given values of m and n, the first mode of vibration with $u_3 \neq 0$ is called the flexural mode and the remaining infinitely many modes are termed the thickness modes of vibration. Flexural modes of vibration are usually predominant in a thin plate but thickness modes of vibration may have lower frequencies than the flexural modes for a thick plate.

Given the plate thickness h, for each combination of values of m and n, the transcendental Eq. (11) can be solved numerically for f and hence ω by using the Newton–Raphson method. Eq. (11) has infinitely many roots; for each root the corresponding mode shape is given by Eq. (7). Thus for each combination of m and n there are infinitely many thickness modes of vibration.

4. Results

We assume that the plate is made of a rubberlike material. Material properties of rubber, taken from the website www.efunda.com, are $E = 1$ MPa, $\rho = 1000\,\text{kg/m}^3$, $\nu = 0.5$, where E is Young's modulus and ν Poisson's ratio. For a square plate and $(m,n) = (1,0),\ (1,1),\ (2,0)$, and $(2,1)$, Table 1 contains a list of the first

Table 2
First 10 natural frequencies of a simply supported square plate made of an incompressible material computed from the analytical solution

$h/L_x = 1/4$				$h/L_x = 1/8$			
Number	(m,n)	Mode	Ω Frequency	Number	(m,n)	Mode	Ω Frequency
1	(1,1)	o	0.577	1	(1,1)	o	0.167
2	(1,0), (0,1)	i	0.785	2	(2,1), (1,2)	o	0.386
3	(1,1)	i	1.111	3	(1,0), (0,1)	i	0.393
4	(2,1), (1,2)	o	1.186	4	(1,1)	i	0.555
5	(2,0), (0,2)	i	1.571	5	(2,2)	o	0.577
6	(2,2)	o	1.656	6	(3,1), (1,3)	o	0.694
7	(2,1), (1,2)	i	1.756	7	(2,0), (0,2)	i	0.785
8	(3,1), (1,3)	o	1.925	8	(3,2), (2,3)	o	0.855
9	(1,1)	i	2.111	9	(2,1), (1,2)	i	0.878
10	(2,2)	i	2.221	10	(1,1)	i	1.097
$h/L_x = 1/12$				$h/L_x = 1/20$			
Number	(m,n)	Mode	Ω Frequency	Number	(m,n)	Mode	Ω Frequency
1	(1,1)	o	0.077	1	(1,1)	o	0.028
2	(2,1), (1,2)	o	0.184	2	(2,1), (1,2)	o	0.069
3	(1,0), (0,1)	i	0.262	3	(2,2)	o	0.109
4	(2,2)	o	0.284	4	(3,1), (1,3)	o	0.135
5	(3,1), (1,3)	o	0.347	5	(1,0), (0,1)	i	0.157
6	(1,1)	i	0.37	6	(3,2), (2,3)	o	0.173
7	(3,2), (2,3)	o	0.437	7	(1,1)	i	0.222
8	(2,0), (0,2)	i	0.524	8	(4,1), (1,4)	o	0.223
9	(4,1), (1,4)	o	0.55	9	(3,3)	o	0.234
10	(3,3)	o	0.577	10	(4,2), (2,4)	o	0.258

three non-dimensional natural frequencies Ω, defined by $\Omega = \omega h\sqrt{\rho/\mu}$ and constants $C_1, C_2, \ldots, C_6$ that determine the corresponding thickness mode shapes. The notations "o" and "i" represent, respectively, the out-of-plane modes with $u_3(x_1, x_2, x_3 = h/2) \neq 0$ for some x_1 and x_2, and the in-plane modes with $u_3(x_1, x_2, x_3 = h/2) = 0$ for every x_1 and x_2. Note that for $\Omega = 2.111$ and 3.099 corresponding to $(m, n) = (1, 1)$ and $(2, 1)$, respectively, constants C_3 and C_4 in Table 1 are complex. However, displacement and pressure amplitudes computed from Eq. (7) are real as illustrated in plots included in Fig. 1.

For $(m, n) = (1, 0)$ and $(1, 1)$, Fig. 1a–d exhibit through-the-thickness variation of $U_1^{mn}, U_2^{mn}, U_3^{mn}$ and P^{mn}, which have been normalized by their corresponding values on plate's bottom surface. Note that some through-the-thickness variations for the considered mode shapes are not displayed in the Figure because their values on the plate's bottom surface equal zero. For example, in Fig. 1c there is no through-the-thickness variation of the normalized U_3^{10}. It is clear from the plot of $U_3^{11}(z)$ in Fig. 1c that plate theories that assume uniform lateral displacement or deflection through the plate thickness will not capture this thickness mode of vibration. From Fig. 1a, we see that U_3^{11} for $\Omega = 2.111$ varies almost linearly through the plate thickness, but that for $\Omega = 0.577$ has a parabolic variation. For $\Omega = 0.577$, U_1^{11} and U_2^{11} variations are close to a polynomial of degree one in z; thus the first-order shear deformation theory can approximately capture this mode of vibration. On the other hand, this is not the case for $\Omega = 2.111$, for which through-the-thickness variations of U_1^{11} and U_2^{11} are not first-order polynomials in z. The through-the-thickness variation of P^{11} corresponding to $\Omega = 0.577$ is linear but it is parabolic for $\Omega = 2.111$. For mode shapes associated with $\{C_1, C_2, \ldots, C_6\} = \{0, 0, 0, 0, 0.707, 0.707\}$, $P^{mn} \equiv 0$ for all values of m and n. We note that the boundary condition of null normal traction on the top and the bottom surfaces of the plate requires that $p = \mu u_{3,3}$ there. Thus P^{11} need not always vanish on these surfaces as $u_{3,3} \neq 0$ on the top and the bottom surfaces of the plate (see Fig. 1c). Of course, $u_{3,3}$ can vanish on the top and the bottom surfaces of the plate without being identically zero through the plate thickness. The nonlinear variations of U_1^{mn}, U_2^{mn} and U_3^{mn} through the plate thickness suggest that a higher-order plate theory such as that proposed by Batra and Vidoli [10] is needed to

Table 3
First 10 natural frequencies of a simply supported rectangular plate ($L_x = 2L_y$) computed from the analytical solution

$h/L_x = 1/4$				$h/L_x = 1/8$			
Number	(m, n)	Mode	Ω Frequency	Number	(m, n)	Mode	Ω Frequency
1	(1,0)	i	0.785	1	(1,1)	o	0.385
2	(1,1)	o	1.186	2	(1,0)	i	0.393
3	(0,1)	i	1.571	3	(2,1)	o	0.577
4	(2,0)	i	1.571	4	(0,1)	i	0.785
5	(2,1)	o	1.656	5	(2,0)	i	0.785
6	(1,1)	i	1.756	6	(3,1)	o	0.855
7	(2,1)	i	2.221	7	(1,1)	i	0.878
8	(3,1)	o	2.283	8	(1,2)	o	1.051
9	(3,0)	i	2.356	9	(2,1)	i	1.111
10	(1,2)	o	2.702	10	(3,0)	i	1.178
$h/L_x = 1/12$				$h/L_x = 1/20$			
Number	(m, n)	Mode	Ω Frequency	Number	(m, n)	Mode	Ω Frequency
1	(1,1)	o	0.184	1	(1,1)	o	0.069
2	(1,0)	i	0.262	2	(2,1)	o	0.109
3	(2,1)	o	0.284	3	(1,0)	i	0.157
4	(3,1)	o	0.437	4	(3,1)	o	0.173
5	(0,1)	i	0.524	5	(1,2)	o	0.223
6	(2,0)	i	0.524	6	(2,2)	o	0.258
7	(1,2)	o	0.55	7	(4,1)	o	0.258
8	(1,1)	i	0.585	8	(0,1)	i	0.314
9	(2,2)	o	0.63	9	(2,0)	i	0.314
10	(4,1)	o	0.63	10	(3,2)	o	0.316

capture all modes of vibration. The minimum order of the plate theory needed to find all frequencies depends upon the aspect ratio of the plate, and boundary conditions at the edges. The compatible higher-order plate theory proposed in [10] employs a mixed variational principle [11].

In Table 2 are given the first ten analytical natural frequencies, for a simply supported square plate made of a homogeneous and isotropic rubberlike material. The plate is usually called thin when $h/L_x < 0.1$. It is clear that, among the first ten modes of vibration, a thick plate ($h/L_x = 1/4$) has several in-plane modes of vibration and their number decreases with a decrease in the plate thickness.

For different values of h/L_x, we have given in Table 3 the first ten analytical natural frequencies of a simply supported rectangular plate having $L_x = 2L_y$. As for a square plate the number of out-of-plane modes increases with a decrease in the value of h/L_x. Whereas only pure distortional modes of vibration are admissible in a plate made of an incompressible material, there is no restriction on the modes of vibration that may occur in a plate made of a compressible material.

We note that by following a procedure similar to that outlined here, one can solve analytically equations corresponding to free vibrations of a plate made of an incompressible anisotropic material, and incompressible piezoelectric material, and hybrid plates with one or more layers made of an incompressible material. Constitutive relations for incompressible linear elastic anisotropic materials may be found in Ref. [12].

5. Conclusions

An analytical solution for free vibration of a simply supported rectangular plate made of an incompressible homogeneous linear elastic isotropic material has been obtained. Some frequencies missing in previous analytical solutions for plates made of compressible materials have been identified.

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