



Correspondence

Closure to “The generalized plane strain deformations of thick anisotropic composite laminated plates”

The authors thank Dr. K.P. Soldatos and Mr. S.-L. Liu for the congratulatory remarks. We tabulate below the numerical results for five different cases and hope that these will prove beneficial to researchers and Liu for validating and refining, if necessary, their plate theories and/or solutions obtained by other approximate techniques such as the finite element method. Results derived from the classical laminated plate theory (CLPT) and the first-order shear deformation theory (FSDT) with a shear correction factor of 5/6 are also included in these tables. The upper value in the brackets is computed with the CLPT and the lower one with the FSDT. In each case, there is a uniformly distributed load acting downwards only on the upper surface of the plate (i.e., $\sigma_{33}(x_1, H) = -q_0$) and the material properties are those given in Vel and Batra (2000a). Using the notations of Vel and Batra (2000a), the normalized quantities used in Tables 1–5 are defined below.

$$\begin{aligned} [\hat{u}_1, \hat{u}_2] &= \frac{100E_T H^2}{q_0 L^3} [u_1, u_2], & \hat{u}_3 &= \frac{100E_T H^3 u_3}{q_0 L^4}, \\ [\hat{\sigma}_{11}, \hat{\sigma}_{12}] &= \frac{H^2}{q_0 L^2} [\sigma_{11}, \sigma_{12}], & [\hat{\sigma}_{13}, \hat{\sigma}_{23}] &= \frac{H}{q_0 L} [\sigma_{13}, \sigma_{23}], \\ \hat{\sigma}_{33} &= \frac{\sigma_{33}}{q_0}, & \hat{e} &= \frac{10E_T}{q_0 H} \left(u_3 \left(\frac{L}{2}, H \right) - u_3 \left(\frac{L}{2}, 0 \right) \right). \end{aligned}$$

All layers are of equal thicknesses for the multilayered laminates.

Results for three-dimensional deformations of a rectangular plate subjected to different boundary conditions are given in Vel and Batra (1999). The authors have generalized the Eshelby–Stroh formalism to the analysis of piezoelectric (Vel and Batra, 2000b,c) and thermoelastic laminates (Vel and Batra, 2000d).

After the submission of the final version of the manuscript in January 1999, the authors came across Vlasov’s (1957), and Srinivas and Rao’s (1973) papers. Vlasov considered simply-supported plates and his analysis is similar to that of Pagano (1969). Srinivas and Rao also studied other boundary conditions and their solution technique can be regarded as a special case of the Eshelby–Stroh formalism.

Table 1
Displacements and stresses at specific locations for a homogeneous [0°] graphite-epoxy plate subjected to various boundary conditions

Normalized variable	Clamped-clamped		Cantilever		Clamped–simply supported	
	L/H = 4	L/H = 8	L/H = 4	L/H = 8	L/H = 4	L/H = 8
$\hat{u}_1(0.75L, H)$	-0.409	-0.258	4.316	3.928	4.016	3.928
$\hat{u}_3(0.5L, 0.5H)$	-1.836	-0.572	-7.418	-2.120	-3.481	-2.120
$\hat{\sigma}_{11}(0.5L, H)$	-0.516	-0.330	0.516	0.750	0.674	0.750
$\hat{\sigma}_{11}(0.9L, 0.75H)$	0.021	0.082	0.054	0.015	0.035	0.015
$\hat{\sigma}_{13}(0.75L, 0.5H)$	0.312	0.355	-0.394	-0.375	-0.379	-0.375
$\hat{\sigma}_{13}(0.9L, 0.5H)$	0.457	0.512	-0.207	-0.150	-0.170	-0.150
$\hat{\sigma}_{33}(0.5L, 0.75H)$	-0.817	-0.840	-0.811	-0.844	-0.839	-0.844
$\hat{\sigma}_{33}(0.9L, 0.75H)$	-0.623	-0.685	-0.800	-0.844	-0.813	-0.844
$\hat{\epsilon}$	-4.680	-4.670	-4.693	-4.687	-4.713	-4.691

Table 2
Displacements and stresses at specific locations for a $[0/90^\circ]$ graphite-epoxy laminate subjected to various boundary conditions

Normalized variable	Clamped-clamped		Cantilever		Clamped–simply supported	
	$L/H = 4$	$L/H = 8$	$L/H = 4$	$L/H = 8$	$L/H = 4$	$L/H = 8$
$\hat{u}_1(0.75L, H)$	$\begin{bmatrix} -2.283 \\ -1.456 \end{bmatrix}$	$\begin{bmatrix} -1.734 \\ -1.456 \end{bmatrix}$	$\begin{bmatrix} 32.453 \\ 30.584 \end{bmatrix}$	$\begin{bmatrix} 30.584 \\ 30.584 \end{bmatrix}$	$\begin{bmatrix} -4.702 \\ -2.185 \end{bmatrix}$	$\begin{bmatrix} -2.941 \\ -2.185 \end{bmatrix}$
$\hat{u}_3(0.5L, 0.5H)$	$\begin{bmatrix} -3.212 \\ -3.343 \end{bmatrix}$	$\begin{bmatrix} -1.317 \\ -1.334 \end{bmatrix}$	$\begin{bmatrix} -19.132 \\ -19.329 \end{bmatrix}$	$\begin{bmatrix} -11.293 \\ -13.302 \end{bmatrix}$	$\begin{bmatrix} -4.257 \\ -4.408 \end{bmatrix}$	$\begin{bmatrix} -2.095 \\ -2.116 \end{bmatrix}$
$\hat{\sigma}_{11}(0.5L, H)$	$\begin{bmatrix} -0.131 \\ -0.078 \end{bmatrix}$	$\begin{bmatrix} -0.094 \\ -0.078 \end{bmatrix}$	$\begin{bmatrix} 0.189 \\ 0.234 \end{bmatrix}$	$\begin{bmatrix} 0.234 \\ 0.234 \end{bmatrix}$	$\begin{bmatrix} -0.191 \\ -0.140 \end{bmatrix}$	$\begin{bmatrix} -0.139 \\ -0.124 \end{bmatrix}$
$\hat{\sigma}_{11}(0.75L, 0)$	$\begin{bmatrix} 0.341 \\ 0.179 \end{bmatrix}$	$\begin{bmatrix} 0.227 \\ 0.179 \end{bmatrix}$	$\begin{bmatrix} -0.353 \\ -0.538 \end{bmatrix}$	$\begin{bmatrix} -0.538 \\ -0.538 \end{bmatrix}$	$\begin{bmatrix} 1.336 \\ 1.076 \end{bmatrix}$	$\begin{bmatrix} 1.147 \\ 1.076 \end{bmatrix}$
$\hat{\sigma}_{13}(0.25L, 0.25H)$	$\begin{bmatrix} -0.509 \\ -0.576 \end{bmatrix}$	$\begin{bmatrix} -0.561 \\ -0.576 \end{bmatrix}$	$\begin{bmatrix} -1.589 \\ -1.729 \end{bmatrix}$	$\begin{bmatrix} -1.729 \\ -1.729 \end{bmatrix}$	$\begin{bmatrix} -0.717 \\ -0.807 \end{bmatrix}$	$\begin{bmatrix} -0.827 \\ -0.847 \end{bmatrix}$
$\hat{\sigma}_{13}(0.75L, 0.25H)$	$\begin{bmatrix} 0.509 \\ 0.576 \end{bmatrix}$	$\begin{bmatrix} 0.561 \\ 0.576 \end{bmatrix}$	$\begin{bmatrix} -0.602 \\ -0.576 \end{bmatrix}$	$\begin{bmatrix} -0.576 \\ -0.576 \end{bmatrix}$	$\begin{bmatrix} 0.332 \\ 0.346 \end{bmatrix}$	$\begin{bmatrix} 0.288 \\ 0.305 \end{bmatrix}$
$\hat{\sigma}_{33}(0.5L, 0.75H)$	$\begin{bmatrix} -0.938 \\ -0.948 \end{bmatrix}$	$\begin{bmatrix} -0.947 \\ -0.948 \end{bmatrix}$	$\begin{bmatrix} -0.935 \\ -0.948 \end{bmatrix}$	$\begin{bmatrix} -0.948 \\ -0.948 \end{bmatrix}$	$\begin{bmatrix} -0.940 \\ -0.948 \end{bmatrix}$	$\begin{bmatrix} -0.947 \\ -0.948 \end{bmatrix}$
\hat{e}	-4.855	-1.913	-8.887	-17.752	-4.150	0.334

Table 3
Displacements and stresses at specific locations for a [45/ -45°] graphite-epoxy laminate subjected to various boundary conditions

Normalized variable	Clamped-clamped		Cantilever		Clamped—simply supported	
	L/H = 4	L/H = 8	L/H = 4	L/H = 8	L/H = 4	L/H = 8
$\hat{u}_1(0.75L, H)$	-2.014	-1.619	34.711	33.994	-3.380	-2.211
$\hat{u}_2(0.75L, H)$	-0.952	-0.762	16.360	15.997	-0.330	0.074
$\hat{u}_3(0.5L, 0.5H)$	-3.580	-1.080	-25.919	-18.346	-4.626	-1.869
$\hat{\sigma}_{11}(0.75L, H)$	-0.175	-0.093	0.211	0.279	-0.636	-0.523
$\hat{\sigma}_{12}(0.75L, 0)$	0.107	0.065	-0.150	-0.195	0.480	0.397
$\hat{\sigma}_{13}(0.75L, 0.25H)$	0.321	0.327	-0.328	-0.327	0.224	0.207
$\hat{\sigma}_{23}(0.25L, 0.25H)$	-0.180	-0.195	-0.553	-0.584	-0.237	-0.266
$\hat{\sigma}_{23}(0.75L, 0.25H)$	0.180	0.195	-0.197	-0.195	0.115	0.123
$\hat{\sigma}_{33}(0.5L, 0.75H)$	-0.799	-0.798	-0.799	-0.798	-0.803	-0.798
$\hat{\epsilon}$	-4.680	-4.670	-4.736	-4.727	-4.742	-4.726

Table 4
Displacements and stresses at specific locations for a $[0/90/0^\circ]$ graphite-epoxy laminate subjected to various boundary conditions

Normalized variable	Clamped-clamped		Cantilever		Clamped–simply supported	
	$L/H = 4$	$L/H = 8$	$L/H = 4$	$L/H = 8$	$L/H = 4$	$L/H = 8$
$\hat{u}_1(0.75L, H)$	-0.569 -0.194	-0.341 -0.194	4.923 4.073	4.264 4.073	-1.353 -1.064	-0.736 -0.291
$\hat{u}_3(0.5L, 0.5H)$	-2.576 -2.473	-0.833 -0.715	-10.321 -9.229	-4.439 -3.956	-3.093 -2.808	-1.096 -0.259
$\hat{\sigma}_{11}(0.5L, H)$	-0.717 -0.259	-0.427 -0.259	0.400 0.778	0.592 0.778	-1.078 -0.595	-0.651 -0.389
$\hat{\sigma}_{11}(0.5L, 0.4H)$	-0.013 0.002	0.002 0.002	-0.020 -0.006	-0.011 -0.006	-0.012 0.005	0.003 0.004
$\hat{\sigma}_{13}(0.25L, 0.15H)$	-0.323 -0.198	-0.243 -0.198	-0.862 -0.595	-0.684 -0.595	-0.398 -0.245	-0.326 -0.297
$\hat{\sigma}_{13}(0.75L, 0.5H)$	0.229 0.347	0.306 0.347	-0.379 -0.347	-0.356 -0.347	0.218 0.266	0.208 0.174
$\hat{\sigma}_{33}(0.5L, 0.75H)$	-0.767 -0.838	-0.823 -0.838	-0.751 -0.838	-0.822 -0.838	-0.786 -0.838	-0.828 -0.838
\hat{e}	-4.686	-4.670	-4.704	-4.704	-4.726	-4.695

Table 5
Displacements and stresses at specific locations for a [45/ - 45/45°] graphite-epoxy laminate subjected to various boundary conditions

Normalized variable	Clamped-clamped		Cantilever		Clamped—simply supported	
	L/H = 4	L/H = 8	L/H = 4	L/H = 8	L/H = 4	L/H = 8
$\hat{u}_1(0.75L, H)$	-1.958 [-0.656 -1.322]	-1.214 [-0.656 -0.945]	17.059 [13.784 15.309]	14.521 [13.784 14.133]	-3.394 [-0.985 -2.924]	-1.949 [-0.985 -1.691]
$\hat{u}_2(0.75L, H)$	0.886 [0.000 0.853]	0.412 [0.000 0.370]	-2.408 [0.000 -1.956]	-0.551 [0.000 -0.447]	0.478 [0.000 0.438]	0.219 [0.000 0.192]
$\hat{u}_3(0.5L, 0.5H)$	-3.708 [-0.438 -3.488]	-1.466 [-0.438 -1.279]	-19.019 [-7.439 -17.238]	-10.824 [-7.439 -10.067]	-4.697 [-0.875 -4.352]	-2.105 [-0.875 -1.865]
$\hat{\sigma}_{11}(0.5L, H)$	-0.452 [-0.250 -0.231]	-0.320 [-0.250 -0.243]	0.581 [0.750 0.767]	0.680 [0.750 0.757]	-0.703 [-0.375 -0.478]	-0.495 [-0.375 -0.410]
$\hat{\sigma}_{12}(0.85L, 0)$	-0.098 [-0.099 -0.072]	-0.091 [-0.099 -0.096]	0.021 [-0.057 -0.115]	-0.027 [-0.057 -0.080]	0.348 [0.262 0.262]	0.266 [0.266 0.234]
$\hat{\sigma}_{13}(0.75L, 0.5H)$	0.378 [0.375 0.428]	0.384 [0.375 0.391]	-0.375 [-0.362 -0.371]	-0.372 [-0.372 -0.372]	0.274 [0.188 0.291]	0.224 [0.188 0.221]
$\hat{\sigma}_{23}(0.25L, 0.25H)$	-0.156 [-0.237 -0.161]	-0.206 [-0.237 -0.215]	-0.532 [-0.711 -0.550]	-0.647 [-0.711 -0.666]	-0.215 [-0.355 -0.226]	-0.300 [-0.355 -0.315]
$\hat{\sigma}_{23}(0.75L, 0.5H)$	-0.011 [0.246 0.082]	0.148 [0.246 0.198]	-0.307 [-0.246 -0.285]	-0.266 [-0.246 -0.254]	-0.025 [0.123 0.031]	0.073 [0.123 0.100]
$\hat{\sigma}_{33}(0.5L, 0.25H)$	-0.179 [-0.156 -0.148]	-0.160 [-0.156 -0.155]	-0.182 [-0.156 -0.146]	-0.160 [-0.156 -0.155]	-0.176 [-0.156 -0.148]	-0.159 [-0.156 -0.155]
$\hat{\epsilon}$	-4.679	-4.670	-4.732	-4.729	-4.739	-4.726

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